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BIFURCATION SET AND INTEGRAL MANIFOLDS OF THE PROBLEM CONCERNING THE MOTION OF A RIGID BODY IN A LINEAR FORCE FIELD

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An integrable problem on the motion of a rigid body, fixed at the center of mass, under the action of a linear force field, being the classical approximation for the central Newtonian field, is studied. The bifurcation set, viz., the collection of critical values of the integral mapping, is found. The types of integral manifolds in nondegenerate cases are indicated.

1. Statement of the problem. Let a rigid body with a fixed center of mass O be placed in a force field induced by an attracting material point P. To the body we attach a cartesian coordinate system with axes directed along the principal axes of inertia of the body. The corresponding moments of inertia are denoted by A, B, C. The ellipsoid of inertia is taken to be triaxial and, without loss of generality, it is assumed that

$$A < B < C \tag{1.1}$$

We introduce the unit vector

$$\nu = \overrightarrow{OP} \,/\, |OP| \tag{1.2}$$

fixed in space, and we denote its components in the moving axes by

$$\nu_1,\ \nu_2,\ \nu_3 \tag{1.3}$$

Let

$$p,\ q,\ r \tag{1.4}$$

be the components in those same axes of the body's instantaneous angular velocity vector. By virtue of the relation

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1 \tag{1.5}$$

the space of variables (1.3) and (1.4) is a five-dimensional manifold $M^5 = S^2 \times \mathbf{R}^3$. The dynamic system on M^5, describing the rigid body's motion, is specified by the Euler – Poisson equations

$$A\frac{dp}{dt} = (B - C)(qr - \varepsilon^2\nu_2\nu_3), \quad \frac{d\nu_1}{dt} = r\nu_2 - q\nu_3 \tag{1.6}$$

$$(ABC,\ pqr,\ 123)$$

In the expansion of the potential energy we have retained only the terms quadratic in variables (1.3), so that $U = {}^1/_2\,(A\nu_1^2 + B\nu_2^2 + C\nu_3^2)\,\varepsilon^2$ (parameter ε depends upon the distance OP and the gravitational constant). We note that parameter ε

can be given any nonzero value by choosing appropriate measurement units for variables (1.4) and time t. In what follows it is convenient to take $\varepsilon^2 = 1/(ABC)$.

Equations (1.6), in addition to the classical area and energy integrals

$$J = Ap\nu_1 + Bq\nu_2 + Cr\nu_3 \tag{1.7}$$

$$H = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + \frac{1}{2}\left(\frac{\nu_1^2}{BC} + \frac{\nu_2^2}{CA} + \frac{\nu_3^2}{AB}\right) \tag{1.8}$$

admit of the quadratic integral first mentioned by Clebsch in the mathematically equivalent problem [1]

$$K = \frac{1}{2}(A^2p^2 + B^2q^2 + C^2r^2) - \frac{1}{2}\left(\frac{\nu_1^2}{A} + \frac{\nu_2^2}{B} + \frac{\nu_3^2}{C}\right) \tag{1.9}$$

System (1.6) with integrals (1.8) and (1.9) admit of a simultaneous change of sign in the quantities p, q, ν_3. The sign of the constant area

$$J = s \tag{1.10}$$

reverses, and, hence, it suffices to study the case $s \geqslant 0$.

We pass from the variables (1.4) to the new variables $\nu_1^{\cdot}, \nu_2^{\cdot}, \nu_3^{\cdot}$ by the formulas

$$\nu_1^{\cdot} = r\nu_2 - q\nu_3,\ \nu_2^{\cdot} = p\nu_3 - r\nu_1,\ \nu_3^{\cdot} = q\nu_1 - p\nu_2 \tag{1.11}$$

Such a change is invertible on set (1.10) and the inverses are determined by the Kolosov formulas [2]

$$p = \frac{C\nu_3\nu_2^{\cdot} - B\nu_2\nu_3^{\cdot} + s\nu_1}{A\nu_1^2 + B\nu_2^2 + C\nu_3^2}, \quad q = \frac{A\nu_1\nu_3^{\cdot} - C\nu_3\nu_1^{\cdot} + s\nu_2}{A\nu_1^2 + B\nu_2^2 + C\nu_3^2} \tag{1.12}$$

$$r = \frac{B\nu_2\nu_1^{\cdot} - A\nu_1\nu_2^{\cdot} + s\nu_3}{A\nu_1^2 + B\nu_2^2 + C\nu_3^2}$$

By (1.11), $\nu_1\nu_1^{\cdot} + \nu_2\nu_2^{\cdot} + \nu_3\nu_3^{\cdot} = 0$, so that the space of variables (1.3) and (1.11) is a tangent lamination of unit sphere (1.5); subsequently denoted $T(S^2)$.

We set $a = 1/A,\ b = 1/B,\ c = 1/C,\ D = bc\nu_1^2 + ca\nu_2^2 + ab\nu_3^2$. Under substitution (1.12) the functions (1.8) and (1.9) become

$$H = \frac{1}{2D}(a\nu_1^{\cdot 2} + b\nu_2^{\cdot 2} + c\nu_3^{\cdot 2} + abcs^2) + \frac{D}{2} \tag{1.13}$$

$$\begin{aligned} K = \frac{1}{2D^2}\{&a^2\nu_1^{\cdot 2} + b^2\nu_2^{\cdot 2} + c^2\nu_3^{\cdot 2} - (a\nu_1\nu_1^{\cdot} + b\nu_2\nu_2^{\cdot} + c\nu_3\nu_3^{\cdot})^2 + \\ &2s[a^2(b-c)\nu_2\nu_3\nu_1^{\cdot} + b^2(c-a)\nu_3\nu_1\nu_2^{\cdot} + c^2(a-b)\nu_1\nu_2\nu_3^{\cdot}] + \\ &s^2(b^2c^2\nu_1^2 + c^2a^2\nu_2^2 + a^2b^2\nu_3^2)\} - {}^1/_2(a\nu_1^2 + b\nu_2^2 + c\nu_3^2) \end{aligned} \tag{1.14}$$

Substitution (1.12) leads Eqs. (1.6) into a system of second-order equations defining a vector field X on $T(S^2)$. This vector field is Hamiltonian in some symplectic lattice [3] and is generated by Hamiltonian (1.3). In this same lattice the Poisson brackets

$$(H, K) \equiv 0 \tag{1.15}$$

Field X is called a reduced system. The reduced system describes the motion of vector (1.2) in a basis connected with the rigid body and, by the same token, defines the body's motion to within a rotation around the axis $O\nu$.

For each pair $(k, h) \in \mathbf{R}^2$, by $I_{k,h}$ we denote a subset in $T(S^2)$, delineated by the relations

$$H = h, \quad K = k \tag{1.16}$$

Obviously, $I_{k,h}$ is an integral surface of the reduced system and is a smooth manifold of dimension 2 for almost all (k, h).

A further problem is to find the set $\Sigma \subset \mathbf{R}^2$ of points (k, h) such that the topological type of $I_{k,h}$ is changed on passing through these points, and to compute this type for the remaining points of $\mathbf{R}^2$. More rigorously, we define the bifurcation set Σ as the set of points $(k, h) \in \mathbf{R}^2$ over which the mapping

$$I = K \times H \colon T(S^2) \to \mathbf{R}^2 \tag{1.17}$$

is not locally trivial. Similarly [4] it can be shown that Σ coincides with the set of critical values of (1.17); if $(k_0, h_0) \in \mathbf{R}^2 \setminus \Sigma$, then for (k, h) close to (k_0, h_0) the manifold $I_{k,h}$ is diffeomorphic with I_{k_0,h_0} smoothly with respect to k and h. In particular, the differentiable type of $I_{k,h}$ is preserved in the connected component of $\mathbf{R}^2 \setminus \Sigma$. According to (1.15) the Arnol'd—Liouville theorem [5] is applicable and, consequently, for $(k, h) \in \mathbf{R}^2 \setminus \Sigma$ either $I_{k,h} = \phi$ or each connected component of $I_{k,h}$ is diffeomorphic with a two-dimensional torus the motion on which is conditionally periodic, so that it remains to find only the connectivity of $I_{k,h}$ in the noncritical cases.

2. Analytic specification of the bifurcation set. On sphere (1.5) we introduce the local coordinates

$$\xi = \int_b^\lambda \tau^{1/2} f^{-1/2}(\tau)\, d\tau, \quad \eta = \int_c^\mu \tau^{1/2} g^{-1/2}(\tau)\, d\tau \tag{2.1}$$

$$f(\tau) = (a - \tau)(b - \tau)(c - \tau), \quad g(\tau) = -f(\tau)$$

$$\nu_1^2 = \frac{(a-\lambda)(a-\mu)}{(a-b)(a-c)}, \quad \nu_2^2 = \frac{(\lambda-b)(b-\mu)}{(a-b)(b-c)}, \quad \nu_3^2 = \frac{(\lambda-c)(\mu-c)}{(a-c)(b-c)}$$

where λ and μ are the elliptic Kolosov variables. By (1.1), $c \leqslant \mu \leqslant b \leqslant \lambda \leqslant a$.

Notes. 1°. The functions $\lambda(\xi)$ and $\mu(\eta)$, being the inverses of the integrals in (2.1), occur in the subsequent exposition. For brevity the argument of these functions is omitted.

2°. Quantities (2.1) are local coordinates on S^2 everywhere except the points $\lambda = \mu = b$ (although the uniqueness is lost upon passing through the coordinate sections, but this does not affect the search for the critical values of (1.17)). The latter points must be investigated separately. Such an investigation is not carried out here since it turns out that the resulting relations are included in the general relation found under the condition $\lambda \neq \mu$.

3°. It can be shown that if $z \in T(S^2)$ is a critical point of mapping (1.17), then a tangent vector $z' \in T(S^2)$ exists, positioned in the first octant of sphere (1.5) and also critical for (1.17), and $I(z) = I(z')$. Because of this all the computations that follow are carried out for points in the first octant.

4°. Parameter s is taken to be strictly positive. The computations simplify significantly when $s = 0$. As we mention below, we restrict ourselves to finding how the corresponding set Σ is obtained from the general case as $s \to 0$.

We express functions (1.13) and (1.14) in variables (2.1) (see Notes 1° and 3°)

$$2H = \frac{\lambda - \mu}{4\lambda\mu} (\xi^{\cdot 2} + \eta^{\cdot 2}) + \frac{abcs^2}{\lambda\mu} + \lambda\mu \tag{2.2}$$

$$2K = \frac{\lambda - \mu}{4\lambda^2\mu^2} (\lambda\xi^{\cdot 2} + \mu\eta^{\cdot 2}) - \frac{s}{\lambda^2\mu^2} [\lambda \sqrt{\lambda g(\mu)}\, \xi^{\cdot} - \mu \sqrt{\mu f(\lambda)}\, \eta^{\cdot}] + \frac{s^2}{\lambda^2\mu^2} [(bc + ca + ab)\lambda\mu - abc(\lambda + \mu)] + \lambda + \mu - (a + b + c) \tag{2.3}$$

Let us consider the function

$$H - \sigma K \tag{2.4}$$

with an undetermined Lagrange multiplier (the undetermined multiplier before H is taken as unity since all critical points of function K prove to be critical also for H). The critical points of (1.17) coincide with those of (2.4) found for all possible values of σ. Equating the gradient of (2.4) with respect to ξ, η, $\xi^{\cdot}$, $\eta^{\cdot}$ to zero, we obtain

$$(\sigma - \mu)(\lambda - \mu)\, \xi^{\cdot} = 2 s\sigma \sqrt{\lambda g(\mu)} \tag{2.5}$$
$$(\lambda - \sigma)(\lambda - \mu)\, \eta^{\cdot} = 2 s\sigma \sqrt{\mu f(\lambda)}$$

$$\left\{\frac{\sigma - \mu}{4\lambda^2\mu} \xi^{\cdot 2} - \frac{\lambda\mu + (\lambda - 2\mu)\sigma}{4\lambda^3\mu} \eta^{\cdot 2} + \frac{s\sigma \sqrt{g(\mu)}}{2\lambda\mu^2 \sqrt{\lambda}} \xi^{\cdot} - \frac{2s\sigma \sqrt{f(\lambda)}}{\lambda^3 \sqrt{\mu}} \eta^{\cdot} + \frac{s^2}{\lambda^2\mu}\left[abc - (bc + ca + ab)\sigma + \frac{abc\sigma}{\lambda\mu}(\lambda + 2\mu)\right] + \sigma - \mu\right\} \sqrt{f(\lambda)} + \frac{s\sigma f'(\lambda)}{2\lambda^2 \sqrt{\mu}} \eta^{\cdot} = 0 \tag{2.6}$$

$$\left\{\frac{\lambda\mu - (2\lambda - \mu)\sigma}{4\lambda\mu^3} \xi^{\cdot 2} + \frac{\lambda - \sigma}{4\lambda\mu^2} \eta^{\cdot 2} + \frac{2s\sigma \sqrt{g(\mu)}}{\mu^3 \sqrt{\lambda}} \xi^{\cdot} - \frac{s\sigma \sqrt{f(\lambda)}}{2\lambda^2\mu \sqrt{\mu}} \eta^{\cdot} + \frac{s^2}{\lambda\mu^2}\left[abc - (bc + ca + ab)\sigma + \frac{abc\sigma}{\lambda\mu}(2\lambda + \mu)\right] - (\lambda - \sigma)\right\} \times \sqrt{g(\mu)} - \frac{s\sigma g'(\mu)}{2\mu^2 \sqrt{\lambda}} \xi^{\cdot} = 0$$

Under the condition $f(\lambda) = g(\mu) = 0$ we obtain the following set of critical points:

$$\lambda = b,\ \mu = c,\ \xi^{\cdot} = \eta^{\cdot} = 0 \tag{2.7}$$
$$\lambda = a,\ \mu = c,\ \xi^{\cdot} = \eta^{\cdot} = 0$$
$$\lambda = a,\ \mu = b,\ \xi^{\cdot} = \eta^{\cdot} = 0$$

(these are none other than the critical points mentioned of the functions K and H themselves). Let

$$f^2(\lambda) + g^2(\mu) \neq 0 \tag{2.8}$$

It is easy to show that

$$\sigma \neq \lambda, \quad \sigma \neq \mu \tag{2.9}$$

and, consequently

$$\xi^{\cdot} = \frac{2s\sigma\sqrt{\lambda g(\mu)}}{(\lambda-\mu)(\sigma-\mu)}, \quad \eta^{\cdot} = \frac{2s\sigma\sqrt{\mu f(\lambda)}}{(\lambda-\mu)(\lambda-\sigma)} \tag{2.10}$$

With a substitution of values (2.1) Eqs. (2.6) reduce to the one equation $(\lambda-\sigma)^2(\sigma-\mu)^2 = s^2 f(\sigma)$. Denoting $F(\sigma) = \sqrt{f(\sigma)}$, from the latter we have

$$(\lambda-\sigma)(\sigma-\mu) = sF(\sigma) \tag{2.11}$$

if $\mu < \sigma < \lambda$ (recall that $s > 0$) and

$$(\lambda-\sigma)(\sigma-\mu) = -sF(\sigma) \tag{2.12}$$

if $\sigma < \mu$ or $\sigma > \lambda$.

By $U^+(\lambda, \mu)$ we denote the set of roots of Eq. (2.11) and by $U^-(\lambda, \mu)$, the set of roots of Eq. (2.12), on which (2.8) and (2.9) are fulfilled and we let

$$U^+ = \bigcup_{\lambda,\mu} U^+(\lambda, \mu), \quad U^- = \bigcup_{\lambda,\mu} U^-(\lambda, \mu)$$

The substitution of (2.10) and (2.11) into functions (2.2) and (2.3) leads to the following parametric equations:

$$\begin{aligned} h &= \frac{\sigma^2}{2} - s\,[F(\sigma) - \sigma F'(\sigma)] \\ k &= \sigma + sF'(\sigma) - \frac{a+b+c}{2} \end{aligned} \tag{2.13}$$

where σ ranges set U^+. We denote the point of set Σ, determined by equalities (2.13), by $\Sigma^+(\sigma)$.

Let us clarify the corresponding calculation. Entering values (2.10) into (2.2), we obtain

$$2h = \frac{s^2}{(\lambda-\sigma)^2(\sigma-\mu)^2}\{abc\,[3\sigma^2 - 2(\lambda+\mu)\sigma + \lambda\mu] - (bc+ca+ab) \times$$
$$[2\sigma - (\lambda+\mu)]\,\sigma^2 + (a+b+c)(\sigma^2 - \lambda\mu)\,\sigma^2 - [(\lambda+\mu)\sigma - 2\lambda\mu]\,\sigma^3\} + \lambda\mu$$

The expression within braces can be represented as

$$\left\{(\lambda-\sigma)(\sigma-\mu)\frac{df(\sigma)}{d\sigma} - f(\sigma)\frac{d}{d\sigma}[(\lambda-\sigma)(\sigma-\mu)]\right\}\sigma - f(\sigma)(\lambda-\sigma)(\sigma-\mu)$$

Using the identity

$$\lambda\mu = \sigma^2 - (\lambda-\sigma)(\sigma-\mu) + \sigma\frac{d}{d\sigma}[(\lambda-\sigma)(\sigma-\mu)]$$

and equality (2.11), we arrive at the first relation in (2.13). We compute the values of function (2.3) at points (2.10)

$$2k = \frac{s^2}{(\lambda-\sigma)^2(\sigma-\mu)^2}\left\{(\lambda-\sigma)(\sigma-\mu)\frac{df(\sigma)}{d\sigma} - f(\sigma)\frac{d}{d\sigma}[(\lambda-\sigma)(\sigma-\mu)]\right\} +$$
$$\lambda + \mu - (a+b+c)$$

With due regard to the identity

$$\lambda + \mu = 2\sigma + \frac{d}{d\sigma}[(\lambda - \sigma)(\sigma - \mu)]$$

we find

$$2k = 2\sigma + \frac{s^2}{(\lambda - \sigma)(\sigma - \mu)} \frac{df(\sigma)}{d\sigma} + \left[1 - \frac{s^2 f(\sigma)}{(\lambda - \sigma)^2 (\sigma - \mu)^2}\right] \times$$
$$\frac{d}{d\sigma}[(\lambda - \sigma)(\sigma - \mu)] - (a + b + c)$$

Substitution of (2.11) into the equality obtained leads to the second relation in (2.13).

Similarly it can be shown that to each value $\sigma \in U^-$ there corresponds a point of set Σ, defined by the system

$$h = \frac{\sigma^2}{2} + s[F(\sigma) - \sigma F'(\sigma)] \qquad (2.14)$$
$$k = \sigma - sF'(\sigma) - \frac{a+b+c}{2}$$

and denoted $\Sigma^-(\sigma)$. Finally,

$$\Sigma = \Sigma^+ \cup \Sigma^-, \quad \Sigma^\pm = \bigcup_{\sigma \in U^\pm} \Sigma^\pm(\sigma)$$

Note. Equalities (2.13) and (2.14) are particular cases of system (2.5) in [6]. This system describes the curve of the multiple roots in a problem with separated variables and essentially determines the bifurcation of certain quadratic integrals. For $s \neq 0$ we have not succeeded in separating the variables in the problem being studied in the present paper (*). We can assume that equations similar to (2.5) in [6] always arise in the investigation of mechanical systems admitting of a complete set of quadratic (not necessarily uniform-quadratic) first integrals in involution.

3. **Construction of set Σ for small values of constant area.** We mark three modes of set Σ, corresponding to (2.7)

$$P_a = (\tfrac{1}{2}(s^2 - a), \tfrac{1}{2}(as^2 + bc))$$
$$P_b = (\tfrac{1}{2}(s^2 - b), \tfrac{1}{2}(bs^2 + ca))$$
$$P_c = (\tfrac{1}{2}(s^2 - c), \tfrac{1}{2}(cs^2 + ab))$$

Let us clarify the structure of sets $U^\pm$, taking the quantity $s > 0$ as sufficiently small. In what follows the latter condition is not specially stipulated; the necessary estimates are easily obtained each time. Equation (2.11) has no more than three roots, and they all lie on segment $[b, a]$ (see Fig. 1). We denote the maximum root of (2.11) by $\sigma^*(\lambda, \mu, s)$. We have

$$\sigma^*(\lambda, \mu, 0) = \lambda \qquad (3.1)$$

*) See: Kharlamov, On the separation of variables in the Clebsch problem. Repts. Abstr. Sixth Kazakhstan Interinst. Conf. Math. Mech., Pt. 2: Mechanics. Alma-Ata, 1977.

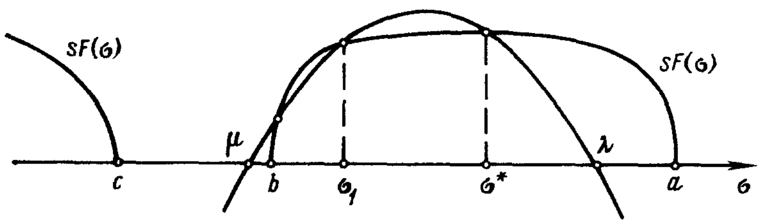

Fig. 1

L e m m a.

$$\max_{\lambda,\mu} \sigma^* (\lambda, \mu, s) = \sigma^* (a, c, s) < a$$

The assertion is trivial if we note that all parabolas $\zeta = (\lambda - \sigma)(\sigma - \mu)$ on the $\zeta\sigma$-plane are congruent. In addition, using (3.1), we find

$$\sigma^* (a, c, s) = a - s^2 \frac{a-b}{a-c} + o(s^2)$$

which is less than a for small s.

The values of $\sigma^* (\lambda, \mu, s)$ wholly fill up the segment $[b, \sigma^* (a, c, s)]$; therefore, the other roots of (2.11), existing when μ is sufficiently close to b, do not add new points into Σ^+.

Finally, $U^+ = (b, \sigma^* (a, c, s)]$. The value $\sigma = b$ is not included in U^+ since either $\mu = b$ or $\lambda = b$, and condition (2.9) comes into play. The value $\sigma = \sigma^* (a, c, s)$ is not formally contained in U^+ since (2.8) is violated. However, making use of equalities (2.5) and the trivial inequality $f(\sigma^* (a, c, s)) \neq 0$, we obtain

$$\lim_{\sigma \nearrow \sigma^*(a,c,s)} \Sigma^+ (\sigma) = P_b \in \Sigma \tag{3.2}$$

Similar limits are computed below without additional stipulations.

Let us consider Eq. (2.12). It has exactly one root $\sigma_* (\lambda, \mu, s)$ on the half-open interval $(b, a]$ and is the larger of the two roots

$$\sigma_0 (\lambda, \mu, s) \leqslant \sigma^\circ (\lambda, \mu, s) \tag{3.3}$$

in the domain $\sigma \leqslant c$. As in the preceding case, it can be shown that the values of (3.3) wholly fill up the segment $[\sigma_0 (b, c, s), c]$, while $\sigma_* (\lambda, \mu, s)$ varies within the limits $b < \sigma_* (b, c, s) \leqslant \sigma_* (\lambda, \mu, s) \leqslant a$. Therefore

$$U^- = [\sigma_0 (b, c, s), c) \bigcup [\sigma_* (b, c, s), a)$$

The values $\sigma = c, \sigma = a$, as well as $\sigma = b$, being roots of (2.12) when $\lambda = b$, have been excluded in accordance with (2.9). The values $\sigma = \sigma_0 (b, c, s)$ and $\sigma = \sigma_*(b, c, s)$ are considered to belong to U^- since

$$\lim_{\sigma \searrow \sigma_0(b,c,s)} \Sigma^- (\sigma) = \lim_{\sigma \searrow \sigma_*(b,c,s)} \Sigma^- (\sigma) = P_a \in \Sigma \tag{3.4}$$

We pass on to the construction of Σ. By $\sigma_1 (a, b, s)$ we denote the root of Eq. (2.11) with $\lambda = a$ and $\mu = b$, differing from $\sigma^* (a, b, s)$. Such a root exists for small s and is unique (see Fig. 1). We have $\Sigma^+ (\sigma_1 (a, b, s)) = \Sigma^+ (\sigma^*$

$(a, b, s)) = P_c$ and, consequently, curve Σ^+ is selfintersecting. On the interval $\sigma_1(a, b, s) < \sigma < \sigma^*(a, b, s)$ the curve Σ^+ has two cusps; we can convince ourselves of this by solving the equation $1 + sF''(\sigma) = 0$ in the form of series in s and comparing the solutions with the expansions of the quantities $\sigma_1(a, b, s)$ and $\sigma^*(a, b, s)$, obtained with due regard to (3.1). The straight line

$$h = b\left(k + \frac{c+a}{2}\right) \tag{3.5}$$

serves as an asymptote to $\Sigma^+(\sigma)$ as $\sigma \searrow b$. The curve $\Sigma^-(\sigma)$ has the node P_a (see (3.4)) and the asymptotes

$$h = a\left(k + \frac{b+c}{2}\right) \quad \text{as} \quad \sigma \nearrow a \tag{3.6}$$

$$h = c\left(k + \frac{a+b}{2}\right) \quad \text{as} \quad \sigma \nearrow c \tag{3.7}$$

The straight lines (3.5) —(3.7) serve as tangents to the parabola

$$h = \frac{1}{2}\left(k + \frac{a+b+c}{2}\right)^2 \tag{3.8}$$

at points

$$k = \frac{b-(c+a)}{2}, \quad k = \frac{a-(b+c)}{2}, \quad k = \frac{c-(a+b)}{2}$$

Finally, we note that the value $\sigma_0(a, c, s) \in U^-$. In this connection, $\Sigma^-(\sigma_0(a, c, s)) = P_b$, so that by virtue of (3.2) the point P_b is common to Σ^+ and Σ^-.

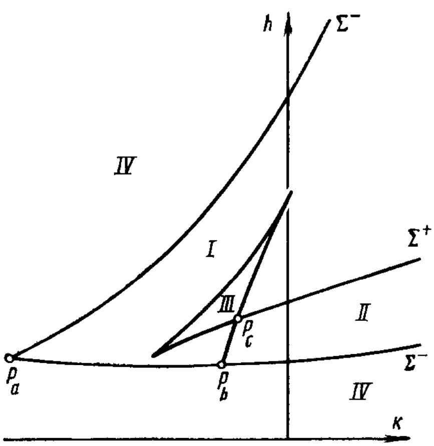

Fig. 2

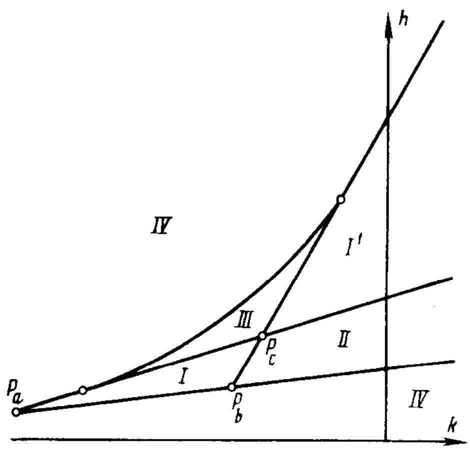

Fig. 3

Summing up what has been said, we obtain the set shown in Fig. 2. Let $s \to 0$. For such a convergence point P_a passes into a point of intersection of straight lines (3.5) and (3.7), point P_b passes into a point of intersection of straight lines (3.6) and (3.7), and point P_c passes into a point of intersection of straight lines (3.5) and (3.6). The segment of curve Σ^+ between the cusps is "spliced" with the segment of parabola (3.8) within the limits

$$\frac{b-(c+a)}{2} \leqslant k \leqslant \frac{a-(b+c)}{2} \tag{3.9}$$

The remainder of Σ degenerates into rays lying on the straight lines (3.5) –(3.7). As a result we obtain the set shown in Fig. 3.

Note. When $s = 0$ equalities (2.13) and (2.14) describe only a part of set Σ (to be precise, a segment of parabola (3.8)). This happens because the remaining critical values of mapping I are reached in this case at points where $f(\lambda) = g(\mu) = 0$ and can be found from Eqs. (2.5) and (2.6).

4. **Structure of the integral manifolds.** As already noted, the connected components of manifolds $I_{k,h}$ are two-dimensional tori when $(k, h) \in \mathbf{R}^2 \setminus \Sigma$. Let us ascertain the connectivity of $I_{k,h}$ for various domains in $\mathbf{R}^2 \setminus \Sigma$. The bifurcation set divides the plane $\mathbf{R}^2$ into four connected domains I –IV when $s \neq 0$ and into five connected domains I, I′, II– IV when $s = 0$. In both cases domain IV contains points with negative coordinates large in absolute value, but since functions K and H are bounded from below, set $I_{k,h}$ is empty for (k, h) belonging to domain IV. For large positive k and h rapid rotations obtain in the problem being studied, so that the reduced system can be interpreted in the standard manner as a perturbation of the reduced system of the Euler –Poinsot problem. In this connection the Clebsch integral splits off from the integral of the modulus of the angular momentum vector. In the Euler –Poinsot case all critical integral manifolds have two connected components [7]. By the Morse theorem [8] this property is preserved under small perturbations of the integral mapping. Consequently, for small $s \neq 0$ we have $I_{k,h} = 2T^2$ in domains I and II. Letting s tend to zero and noticing that the manifolds being examined remain noncritical, we get that $I_{k,h} = 2T^2$ in domains I, I′, II when $s = 0$.

Things become somewhat more complicated in domain III. From considerations akin to the Morse theorem it follows that the connectivity of $I_{k,h}$ is one and the same when $s = 0$ and when $0 < s \ll 1$. We can make use of the fact that manifolds $I_{k,h}$ can be found explicitly when $s = 0$. Indeed, from (1.16), with due regard to (2.2) and (2.3) with $s = 0$, we find

$$2k + a + b + c - \left(\lambda + \frac{2h}{\lambda}\right) = \frac{(\lambda-\mu)^2}{4\lambda^2\mu^2}\zeta^{\cdot 2} \geqslant 0$$

$$\mu + \frac{2h}{\mu} - 2k - (a+b+c) = \frac{(\lambda-\mu)^2}{4\lambda^2\mu^2}\eta^{\cdot 2} \geqslant 0$$

Therefore, the motion takes place in the domain

$$\frac{2h}{\lambda} + \lambda \leqslant 2k + a + b + c \leqslant \frac{2h}{\mu} + \mu \tag{4.1}$$

The condition that (k, h) belongs to domain III is determined by the inequalities (3.9) and

$$\max\left[b\left(k + \frac{c+a}{2}\right),\ a\left(k + \frac{b+c}{2}\right)\right] < h < \frac{1}{2}\left(k + \frac{a+b+c}{2}\right)^2$$

(see Fig. 3); whence we conclude that domain (4.1) consists of two rings

$$b < k_1 - \sqrt{k_1^2 - 2h} \leqslant \lambda \leqslant k_1 + \sqrt{k_1^2 - 2h} < a \tag{4.2}$$

(μ is arbitrary, $2k_1 = 2k + a + b + c$) located symmetrically relative to the section of sphere (1.5) by plane $\nu_1 = 0$. When $s = 0$ the reduced system is natural, and the motion with respect to each of rings (4.2) takes place in two directions. The ring with a fixed direction of motion yields a connected manifold - a two-dimensional torus - in the phase space. Finally, $I_{k,h} = 4T^2$. By virtue of what has been said above, the latter equality holds for (k, h) belonging to domain III and for sufficiently small values of s.

Note. By using the parameteric Eqs. (2.13) and (2.14) the set Σ can be constructed for arbitrary values of the constant of areas. Beginning with some $s_0 > 0$ it turns out to be simpler than the set shown in Fig. 2 (the cusps and the selfintersection of curve Σ^+ disappear). The analysis of the corresponding integral manifolds poses no difficulty. It seems that the greatest interest is in the investigation of the critical integral surfaces since the motions on them are of a nontrivial nature.

In conclusion we take note of [9] in which the bifurcation functions (1.7) and (1.8) are found for the general problem of the motion of a body with a fixed point in a Newtonian field. In this case the integral manifolds are three-dimensional, and it is not possible to lower their dumension because of the absence of the first integral (1.9). The method proposed above carries the investigation of the phase topology through to the end in the case when the center of mass coincides with the fixed point.

REFERENCES

1. Clebsch, R. F. A., Uber die Bewegung eines Körpers in einer Flüssigkeit. Math. Ann., Bd. 3, S. 238 – 262, 1870.

2. Kolosov, G. V., On certain modifications of Hamilton's principle in application to solving problems in rigid body mechanics. S.-Pb., 1903.

3. Kharlamov, M.P., Lowering of orders in mechanical systems with symmetry. In: Rigid Body Mechanics, No. 8. Kiev, "Naukova Dumka", pp. 4 – 18, 1976.

4. Tamarinov, Ia. V., On the investigation of the phase topology of compact configurations with symmetry. Vestn. Mosk. Gos. Univ., Ser. Mat., Mekh., No. 5, 1973.

5. Arnol'd, V. I., On a Liouville theorem touching on integrable problems of dynamics. Sib. Mat. Zh., Vol. 4, No. 2, 1963.

6. Alekseev, V. M., A generalized spatial problem of two fixed centers. Classification of the motions. Biul. Inst. Teor. Astron. Akad. Nauk SSSR, Vol. 10, No. 4, 1965.

7. Kharlamov, M. P., Integral manifolds of the reduced system in the problem of inertial motion of a rigid body with a fixed point. In: Rigid Body Mechanics, No. 8, Kiev, "Naukova Dumka", pp. 18 – 23, 1976.

8. M i l n o r, J., Morse Theory. Moscow, "Mir", 1965.

9. T a m a r i n o v, Ia. V., Patterns of classical integrals of the problem of a rigid body's rotation around a fixed point. Vestn. Mosk. Gos. Univ., Ser. Mat., Mech., No. 6, 1974.

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